DISPERSION OF A PACKET OF PLATES IN THE ATMOSPHERE

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We shall consider the problem of the dispersion of a packet of rigid, homogeneous, greatly elongated rectangular plates dropped from a great height in the atmosphere. When the packet is dropped, sooner or later the phenomenon of autorotation appears in every plate, that is, the plate begins to rotate about its long axis and, as a consequence, acquires a horizontal component of velocity. A packet of such plates will disperse and form a figure in space bounded by the moving plates.

We now pose the problem of the shape of this figure, its position in space and the surface density of the plates as a function of time, on the assumption of zero turbulence in the atmosphere.

We shall assume that the fall of the plates is regular in the sense that the center of gravity of a plate is in one plane and the long axis of the plate is horizontal during all the time it is in motion. The plate is subjected to the force exerted by its own weight and aerodynamic forces. The resistance of the medium is accepted as being proportional to the square of the velocity.

The equation of motion for the steady-state case in the natural coordinate system in the projection on the axis whose axis coincides with the velocity V will be [1]

$$mg\cos\beta = kV^2. \tag{1}$$

Here m is the mass of a plate, β is the angle of inclination of the path of the center of gravity of the plate to the vertical, g is the acceleration of free fall, and k is the drag.

Formula (1) includes the unknown parameters k and β. Making use of reasoning from dimensional analysis [2], we obtain the following expressions for them:

$$k = 2\rho a l \psi\left(\frac{m}{\rho a^3}, \frac{\omega l}{V}, R\right), \tag{2}$$

$$\cos\beta = \varphi\left(\frac{m}{\rho a^3}, \frac{\omega l}{V}, R\right). \tag{3}$$

Here, l is the length of a plate, a is half the width of a plate, ω is the angular velocity of rotation of a plate, ρ is the density of air, R is the Reynolds number, ψ and φ are the unknown functions of three dimensionless parameters. We can retain only one dimensionless combination $m/\rho a^3 \equiv A$. As is known [1], it is approximately true that $V = 2a$. Consequently, the parameter $\omega l/V$ for plates with a given elongation is a constant. We can state in regard to the Reynolds number that the motion depends considerably less on it than on the parameter A.

An approximate expression for the functions ψ and φ depending on the parameter A was derived through processing experimental data on dropping the plates from a certain height. Here, we shall present only the final semiempirical formulas for the velocity and the angle of inclination of the path of the center of gravity to the vertical:

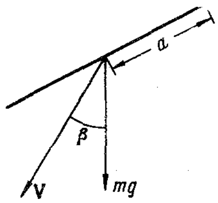
$$V = 2.5\sqrt{(ga^2/l)\,A^{0.92}}, \qquad \cos\beta = 0.54A^{0.093}. \tag{4}$$

The accuracy of formula (4) is higher than 90% with a range of variation in A from 3 to 50.

We note further that the variation in the density of the air with height is approximately expressed by the formula $\rho = \rho_0 \exp(-y)$, where ρ_0 is the density when $y = 0$ and τ = const. By dividing the path of the fall of a plate into segments and considering the density of the air in each of them to be constant, we shall find the rate of fall of a plate V expressed by formula (4).

On the basis of the foregoing, we can state that the law of motion of a single autorotating plate falling through the atmosphere is known to some approximation to reality.

We shall now go on to the basic problem of the dispersion of a packet of plates.



Let a packet containing a very large number N of individual plates be dropped at a certain height h in the air. It is obvious that the phenomenon of autorotation will not begin at the same time for all plates. A certain figure is formed in space which is bounded by dispersing plates which, we shall assume, are flying regularly. It is necessary to determine the form and position in space of this figure, also the surface density of the plates as a function of time, considering the motion of a single plate to be known on the basis of the preceding considerations.

We now solve the problem when each one of the N plates begins to rotate. We select the value of the path traversed by a plate without rotating after it starts its fall as the characteristic parameter, considering the point where the packet was ejected as the origin of coordinates and the y-axis as directed downward. It is obviously impossible to give some concrete value of y and to say that a plate will begin to rotate at this point, as this value will depend on a set of values, often random. Consequently, we can only speak of the probability that a given plate will begin to autorotate in a given interval from y to $y + \Delta y$.

We assume that this probability obeys a Maxwellian distribution. In this case, in an interval from y to $y + \Delta y$, $\Delta N(y)$ of the total N plates will begin to autorotate, $\Delta N(y)$ satisfying the formula

$$\Delta N(y) = 4\pi N(\sigma/\pi)^{3/2} y^2 \exp(-\sigma y^2)\,\Delta y, \tag{5}$$

where σ is an unknown parameter characterizing the distribution and depending, first of all, on the density of the air and the characteristics of the plate.

It is natural to consider that the plates will disperse uniformly in all directions from the vertical. All rotating plates flying in a given direction will move together, since they are the same and will have the same speed defined by (4). However, they begin to autorotate at different times. Consequently the plates flying in a given direction are distributed along a straight line which will form a conical surface. If we take the variation in the density of the air with height, this surface will differ somewhat from a cone. Moreover, it is necessary to note that the surface will be blurred, that is, it will have some thickness on the strength of various random causes.

In the case of ordered, not chaotic ejection of a packet of plates, there will be no axial symmetry since the dispersion of the plates in different directions will no longer be equally probable, but in this formulation of the problem, this question will not be considered, as is also true of the question of the effect of applied currents of air.

We can consider that all plates drop with velocity V_1 before the beginning of rotation, this quantity being equal to the rate of fall in air without rotation. Then, the vertex of the cone falls with the velocity V_1, since nonrotating plates are dropping in it.

Starting with geometric considerations and the fact that the velocity of a rotating plate is V, and the angle of inclination of the path of the center of gravity to the vertical is β, we introduce the following quantities: the radius of the base, and the height and area of the laterial surface of the cone as functions of time.

We shall give, for example, only the value of the lateral surface,

$$S = \pi V t^2 \sqrt{V^2 + V_1^2 - 2VV_1 \cos\beta} \sin\beta. \qquad (6)$$

Knowing the area of the lateral surface of the cone, we can determine the mean surface density of plates as a function of time

$$c = \frac{1}{S} \int_0^y \Delta N(y) \qquad \left(y = \int_0^t V_1\, dt\right). \qquad (7)$$

The local surface density determined in some zone of unit height of the surface of the cone will vary as a function of the law of distribution.

Further, we shall consider the problem of the value of the parameter σ which characterizes the maximum of the distribution, that is, yields that value of $y = 1/\sqrt{\sigma}$ at which the beginning of autorotation of a plate is most probable. The value at which the plate reaches maximum velocity in its fall without rotation will be this value y^*. The best conditions are created at this time for separation of the plate and the beginning of rotation. In this connection, it is necessary to solve the problem of a plate dropping in air without rotation, considering the density to be variable, and to find that value of y^* at which the maximum velocity is reached.

The equation of motion of a plate falling in air without rotation, with variation of density with height taken into consideration, will be

$$m\, dV_1 / dt = mg - k\rho s V_1^2 \qquad (\rho = \rho_0 e^{-\tau y}). \qquad (8)$$

Here s is the area of a plate. The equation is written for the y-axis directed upward, with the origin on the level $\rho = \rho_0$. The initial conditions are $V_1 = 0$, $y = y_1$, that is, the plates are ejected at the height y_1. This equation is reduced to the Bernoulli equation; after solving it, we have

$$V_1 = \left(\frac{2g}{\tau} e^{-u} \int_{u_1}^{u} e^u \frac{du}{u}\right)^{1/2} \qquad \left(u = \frac{2k\rho_0 s}{\tau m} e^{-\tau y}\right). \qquad (9)$$

To determine the position of the maximum of velocity, we derive the relationship

$$\int_{u_1}^{u} \frac{e^u}{u}\, du = \frac{e^u}{u}. \qquad (10)$$

It is required to determine that value of u^* at which (10) is satisfied. Knowing this, we obtain y^*, and from it

$$\sigma = \tau^2 \left(\ln \frac{\tau m u^*}{2k\rho_0 s}\right)^{-2}. \qquad (11)$$

REFERENCES

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2. L. I. Sedov, Similarity Methods and Dimensional Analysis in Mechanics [in Russian], Gostekhizdat, 1954.

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